The typicality approach to thermodynamical relaxation in quantum systems.

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Why typicality?

The traditional view on relaxation / 2nd law of thermodynamcis:

QM: $$\hat{\rho} = |\psi\rangle\langle\psi| \quad \Rightarrow \quad \hat{\rho} = \frac{1}{Z}\mathrm{e}^{-\frac{\hat{H}}{kT}} \, , \, \hat{\rho} = \frac{1}{Z}\hat{\delta}(\hat{H} - E)$$

CM: $$\rho(x,p) = \delta(x - x_0)\delta(p - p_0) \quad \Rightarrow \quad \rho = \frac{1}{Z}\mathrm{e}^{-\frac{H(x,p)}{kT}} \, , \, \rho = \frac{1}{Z}\delta(H(x,p) - E)$$

problems:

CM: invariance of Von Neuman entropy, ergodicity, mixing, etc.

QM: (framework of open quantum systems) large (stationary, broad band) environment, adequate weak coupling, pertinent factorizing initial state, etc.

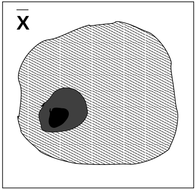
Typicality:

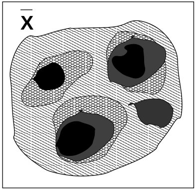
$$\rho = |\psi\rangle\langle\psi| \quad \text{does not evolve into} \quad \hat{\rho} = \frac{1}{Z}\mathrm{e}^{-\frac{\hat{H}}{kT}} \, , \, \hat{\rho} = \frac{1}{Z}\hat{\delta}(\hat{H} - E)$$

$$\text{but} \quad \langle\psi|\hat{A}(t)|\psi\rangle \Rightarrow \approx \frac{1}{Z}\mathrm{Tr}\{\mathrm{e}^{-\frac{\hat{H}}{kT}} \hat{A}\}$$

for very many (all?) $\hat{A}$

The typicality scenario





$\overline{x}$: microstates in state space, AR: accessicle region due to constants of motion, $f(\overline{x})$: considered quantity

- Average: $E_{AR}(f) = \int_{AR} f(\overline{x})\, dV_{\overline{x}}$
- Variance: $V_{AR}(f) = E_{AR}[f^2] - E_{AR}[f]^2$

Typicality: $V_{AR}^{\frac{1}{2}}[f] \leq f_{\max} - f_{\min}$

$\Rightarrow$ relative frequency of stats featuring $f(\overline{x}) \approx E_{AR}(f)$ is high

connection to dynamics possible if $\dot{\overline{x}} = \overline{G}(\overline{x}) \qquad div_{\overline{x}}\overline{G} = 0$

invariance of state space volume, no ergodicity, mixing, etc.

Typicality in QM

basics and non-composite systems state:

$$|\psi\rangle = \sum_n \psi_n \, |n\rangle = \sum_n (\eta_n + i\,\xi_n |n\rangle)$$

$$\overline{x} = \{\eta_n, \xi_n\} \quad : \; \eta_n, \xi_n \; : \text{real cartesian coordianates}$$

dynamics: Schrödinger equation, $\dot{\overline{x}} = \overline{H}(\overline{x})$ $\quad div_{\overline{x}} \overline{H} = 0$

accessible region: $\hat{\Pi}_\alpha$: projective constants of motion (invariant subspaces), e.g., spanned by energy eigenstates, spanned by states featuring equal particle number, etc.

$$\hat{\Pi}_\alpha^2 = \hat{\Pi}_\alpha \quad , \quad \left[\hat{H}, \hat{\Pi}_\alpha\right] = 0 \quad , \quad N_\alpha = \mathrm{Tr}\{\hat{\Pi}_\alpha\} \quad , \quad \mathrm{Tr}\{\hat{\Pi}_\alpha, \hat{\Pi}_\beta\} = N_\alpha \delta_{\beta\alpha}$$

$\mathrm{AR}: \quad \{\langle\psi|\hat{\Pi}_\alpha|\psi\rangle = W_\alpha\}$ occupation probabilities of subspaces conserved

considered quantity : $f(\overline{x}) = \langle\psi|\hat{A}|\psi\rangle$

Hilbertspace average of observables:

$$E_{AR}[f] \equiv [\![\langle\psi|\hat{A}|\psi\rangle]\!]_{AR} = \mathrm{Tr}\{\hat{A}\hat{\Omega}\} \quad , \quad \hat{\Omega} = \sum_{\alpha} \frac{W_\alpha}{N_\alpha}\hat{\Pi}_\alpha$$

Boltzmann state featuring “equal a priori probabilities

Hilbertspace variance of observables: $(\hat{A}_{\alpha\beta} \equiv \hat{\Pi}_\alpha \hat{A} \hat{\Pi}_\beta)$

$$V_{AR}[f] \equiv \Delta_H^2(\langle\hat{A}\rangle) = \sum_{\alpha\beta} \frac{W_\alpha W_\beta}{N_\alpha(N_\beta + \delta_{\alpha\beta})}\left(\mathrm{Tr}\{\hat{A}_{\alpha\beta}\hat{A}^\dagger_{\alpha\beta}\} - \delta_{\alpha\beta}\frac{\mathrm{Tr}\{\hat{A}_{\alpha\alpha}\}^2}{N_\alpha}\right)$$

consider, e.g.: ($\Delta_S^2(\hat{A})$: spectral variance)

$$\langle\psi|\hat{\Pi}_\alpha|\psi\rangle = 1 \quad \Rightarrow \quad \Delta_H^2(\langle\hat{A}\rangle) = \frac{1}{N_\alpha + 1}\Delta_S^2(\hat{A})$$

typicality: requires high dimensional space, bounded spectra

typicality of states: (squared) distance of most states within the AR to some "typical state" is small. $\hat{\Omega}$: canditate for a typical state

$$\text{mean squared distance:} \qquad \Delta^2_H(\hat{\rho}) = [\![\mathrm{Tr}\{(\hat{\rho} - \hat{\Omega})^2\}]\!]_{AR}$$

for non-composite systems

$$\Delta^2_H(\hat{\rho}) = 1 - \mathrm{Tr}\{\hat{\Omega}^2\} \approx 1 \quad \Rightarrow$$

for Boltzmann-type $\hat{\Omega}$'s not small at all.

Conclusion on non-composite systems:
Many observables may relax, but the state does not. There is no increase of Von Neumann entropy.

composite systems: $H = H_S + H_E + V$
projective subspaces:
sys.:$\hat{\Pi}_A : E_A - \frac{1}{2}\delta_A \leq E \leq E_A + \frac{1}{2}\delta_A$, env.: $\hat{\Pi}_B : E_B - \frac{1}{2}\delta_B \leq E \leq E_B + \frac{1}{2}\delta_B$

$$\text{definition of global projective subspaces:} \quad \hat{\Pi}_\alpha = \hat{\Pi}_A \hat{\Pi}_B$$

Microcanonial Scenario: $[H_S, H] = 0$

accessible region $AR: \quad \{\langle\psi|\hat{\Pi}_A|\psi\rangle = W_A\,, \quad \langle\psi|\hat{\Pi}_B|\psi\rangle = W_B\}$

energies in system/environment seprately conserved, no energy in coupling.

Candidate for the typical local system state:

$$\Omega \equiv \sum_A \frac{W_A}{N_A}\hat{\Pi}_A \qquad \text{local Boltzmann state}$$

$\Rightarrow$

mean squared distance: $$\hat{\Delta}^2_H(\hat{\rho}) = \sum_B \frac{W_B^2}{N_B}\left(1 - \sum_A \frac{W_A^2}{N_A^2}\right)$$

scaling of upper bound with subsystem sizes:

$$N_A \rightarrow x\, N_A \qquad N_B \rightarrow x\, N_B \qquad \Rightarrow \qquad \Delta^{2+}_H(\hat{\rho}) \rightarrow \frac{1}{y}\Delta^{2+}_H(\hat{\rho})$$

For large environemnts there is full typicality of state, increase of entropy, etc.

energy exchange scenario

$$\text{total energy subspaces:} \quad \hat{\Pi}_E = \sum_{E_A + E_B \approx E} \hat{\Pi}_A \hat{\Pi}_B$$

$$\text{accessible region} \quad AR: \quad \{\langle\psi|\hat{\Pi}_E|\psi\rangle = W_E\}$$

only total energy is conserved, no energy in the coupling

Candidate for the typical local system state:

$$\Omega \equiv \mathrm{Tr}_{Env}\{\sum_E \frac{W_E}{N_E}\hat{\Pi}_E\}$$

scaling of upper bound with subsystem sizes:

$$N_A \to x\, N_A \qquad N_B \to x\, N_B \qquad \Rightarrow \qquad \Delta_H^{2+}(\hat{\rho}) \to \frac{1}{y}\Delta_H^{2+}(\hat{\rho})$$

For large environemnts there is full typicality of state, increase of entropy, etc.

Canonical Scenario:

What about the standard canonical Gibbs state?

A state density in the environment yielding $N_B \propto \mathrm{e}^{cE}$ can be expected for environments made of weakly interacting subsystems.

In this case one finds fort the typical energy exchange state:

$$\Rightarrow \quad \Omega = \frac{1}{Z}\mathrm{e}^{-cE_A}\hat{\Pi}_A \approx \frac{1}{Z}\mathrm{e}^{-\frac{\hat{H}}{kT}}, \qquad c \approx \frac{1}{kT}$$

Comment on relaxation in composite systems:
All relaxation in composite systems, regardeless of the strength of the interaction, is due to increasing correlations/entanglement.

Dynamical or off-equilibrium typicality

consider more general AR:

$$\mathrm{AR}: \qquad \{\langle\psi|\hat{A}|\psi\rangle = A\,, \quad \langle\psi|\psi\rangle = 1\}$$

$\hat{A}$: any hermitian Observable.

How differently will the various $\langle\psi|\hat{A}|\psi\rangle$ from the AR erolve?

$$\Delta^2_H\left(\langle\psi|\hat{A}(t)|\psi\rangle\right) = [\![\langle\psi|\hat{A}(t)|\psi\rangle^2]\!]_{AR} - [\![\langle\psi|\hat{A}(t)|\psi\rangle]\!]^2_{AR}$$

Hard to answer. But consider:

$$|\phi\rangle = \left(\hat{1} + \frac{d}{(1+d^2)}\hat{A}\right)|\theta\rangle\,, \qquad \mathrm{Tr}\{\hat{A}\} = 0, \qquad \mathrm{Tr}\{\hat{A}^j\} = c_j$$

with $c_2 = 1$ and $\frac{c_j}{N}$ approximately independent of N

accessible region for the $|\theta\rangle$'s: $AR: \quad \{\langle\theta|\theta\rangle = 1\} \Rightarrow$

$$[\![\langle\phi|\phi\rangle]\!]_{AR} = 1, \qquad \Delta_H^2(\langle\phi|\phi\rangle) \propto \frac{1}{N}$$

$$[\![\langle\phi|\hat{A}|\phi\rangle]\!]_{AR} = 2d, \qquad \frac{\Delta_H^2(\langle\phi|\hat{A}|\phi\rangle)}{d^2} \propto \frac{1}{N}$$

N large: almost all $|\phi\rangle$ are from the AR with $A = 2d$!

result for the dynamics of the $|\phi\rangle$'s:

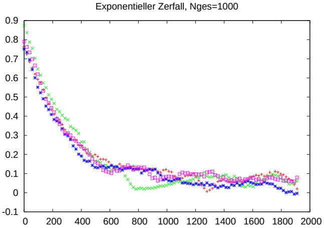
$$\frac{\sqrt{\Delta_H^2(\langle\phi|\hat{A}(t)|\phi\rangle)}}{2d} \leq \frac{1}{\sqrt{N}}$$

$$[\![\langle\phi|\hat{A}(t)|\phi\rangle]\!]_{AR} = \mathrm{Tr}\{\hat{A}\hat{\rho}(t)\}, \qquad \hat{\rho}(0) = \hat{1} + 2d\hat{A}$$

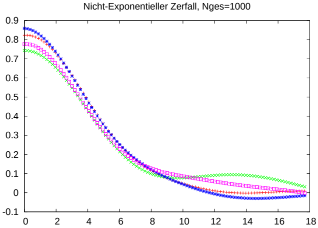
For large N almost all $\langle\psi|\hat{A}(t)|\psi\rangle$ evolve very similar!
The average evolution of all $\langle\psi|\hat{A}(t)|\psi\rangle$ may possibly be computed with projection methods $\Rightarrow$ The inhomogeneity in the NZ-equation may almost always be neglected.

“generic evolutions”: $N = 1000$

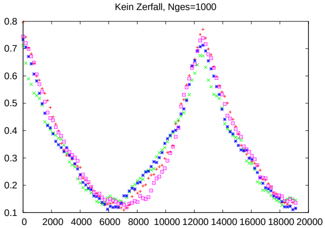
“standard” weak coupling



strong coupling



“pathological” weak coupling



The “take home message”:
According to the typicality approach relaxation is not a necessity, but something that is extrem likely to happen in complex systems.

more information, publications: ask me or visit our webpage.

Many thanks to M. Michel, C. Bartsch, ... and the audience!